IMO 2014
Problems and Solutions

Problem 1. Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + \cdots + a_n}{n} \leq a_{n+1}. \qquad (1)$$

Solution. For $n = 1, 2, \ldots$ define

$$d_n = (a_0 + a_1 + \cdots + a_n) - na_n.$$

The sign of d_n indicates whether the first inequality in (1) holds; i.e., it is satisfied if and only if $d_n > 0$.

Notice that

$$na_{n+1} - (a_0 + a_1 + \cdots + a_n) = (n+1)a_{n+1} - (a_0 + a_1 + \cdots + a_n + a_{n+1}) = -d_{n+1},$$

so the second inequality in (1) is equivalent to $d_{n+1} \leq 0$. Therefore, we have to prove that there is a unique index $n \geq 1$ that satisfies $d_n > 0 \geq d_{n+1}$.

By its definition the sequence $d_1, d_2, \ldots$ consists of integers and we have

$$d_1 = (a_0 + a_1) - 1 \cdot a_1 = a_0 > 0.$$

From

$$d_{n+1} - d_n = \big((a_0 + \cdots + a_n + a_{n+1}) - (n+1)a_{n+1}\big) - \big((a_0 + \cdots + a_n) - na_n\big) = n(a_n - a_{n+1}) < 0$$

we can see that $d_{n+1} < d_n$ and thus the sequence strictly decreases.

Hence, we have a decreasing sequence $d_1 > d_2 > \cdots$ of integers such that its first element d_1 is positive. The sequence must drop below 0 at some point, and thus there is a unique index n, that is the index of the last positive term, satisfying $d_n > 0 \geq d_{n+1}$.

Comment. Omitting the assumption that $a_0, a_1, \ldots$ are integers allows the numbers d_n to be all positive. In such cases the desired n does not exist. This happens for example if $a_n = 2 - \frac{1}{2^n}$ for all integers $n \geq 0$.

Problem 2. Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

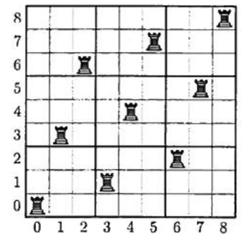
Answer. $\lfloor \sqrt{n-1} \rfloor$.

Solution. Let ℓ be a positive integer. We will show that (i) if $n > \ell^2$ then each peaceful configuration contains an empty $\ell \times \ell$ square, but (ii) if $n \leq \ell^2$ then there exists a peaceful configuration not containing such a square. These two statements together yield the answer.

(i). Assume that $n > \ell^2$. Consider any peaceful configuration. There exists a row R containing a rook in its leftmost cell. Take ℓ consecutive rows with R being one of them. Their union U contains exactly ℓ rooks. Now remove the $n - \ell^2 \geq 1$ leftmost columns from U (thus at least one rook is also removed). The remaining part is an $\ell^2 \times \ell$ rectangle, so it can be split into ℓ squares of size $\ell \times \ell$, and this part contains at most $\ell - 1$ rooks. Thus one of these squares is empty.

(ii). Now we assume that $n \leq \ell^2$ (in particular, we have $\ell \geq 2$). Firstly, we will construct a peaceful configuration with no empty $\ell \times \ell$ square for the case $n = \ell^2$. After that we will modify it to work for smaller values of n.

Let us enumerate the rows from bottom to top as well as the columns from left to right by the numbers $0, 1, \ldots, \ell^2 - 1$. Every cell will be denoted, as usual, by the pair (r, c) of its row and column numbers. Now we put the rooks into all cells of the form $(i\ell + j, j\ell + i)$ with $i, j = 0, 1, \ldots, \ell - 1$ (the picture below represents this arrangement for $\ell = 3$). Since each number from 0 to $\ell^2 - 1$ has a unique representation of the form $i\ell + j$ $(0 \leq i, j \leq \ell - 1)$, each row and each column contains exactly one rook.



Next, we show that each $\ell \times \ell$ square A on the board contains a rook. Consider such a square A, and consider ℓ consecutive rows the union of which contains A. Let the lowest of these rows have number $p\ell + q$ with $0 \leq p, q \leq \ell - 1$ (notice that $p\ell + q \leq \ell^2 - \ell$). Then the rooks in this union are placed in the columns with numbers $q\ell + p$, $(q+1)\ell + p$, $\ldots$, $(\ell - 1)\ell + p$, $p + 1$, $\ell + (p+1)$, $\ldots$, $(q-1)\ell + p + 1$, or, putting these numbers in increasing order,

$$p + 1,\ \ell + (p+1),\ \ldots,\ (q-1)\ell + (p+1),\ q\ell + p,\ (q+1)\ell + p,\ \ldots,\ (\ell - 1)\ell + p.$$

One readily checks that the first number in this list is at most $\ell - 1$ (if $p = \ell - 1$, then $q = 0$, and the first listed number is $q\ell + p = \ell - 1$), the last one is at least $(\ell - 1)\ell$, and the difference between any two consecutive numbers is at most ℓ. Thus, one of the ℓ consecutive columns intersecting A contains a number listed above, and the rook in this column is inside A, as required. The construction for $n = \ell^2$ is established.

It remains to construct a peaceful configuration of rooks not containing an empty $\ell \times \ell$ square for $n < \ell^2$. In order to achieve this, take the construction for an $\ell^2 \times \ell^2$ square described above and remove the $\ell^2 - n$ bottom rows together with the $\ell^2 - n$ rightmost columns. We will have a rook arrangement with no empty $\ell \times \ell$ square, but several rows and columns may happen to be empty. Clearly, the number of empty rows is equal to the number of empty columns, so one can find a bijection between them, and put a rook on any crossing of an empty row and an empty column corresponding to each other.

Comment. One may apply a different argument for part (*i*). In view of the arguments in the last paragraph of the solution, it suffices to deal only with the case $n = \ell^2 + 1$. Notice now that among the four corner cells, at least one is empty. So the rooks in its row and in its column are distinct. Now, deleting this row and column we obtain an $\ell^2 \times \ell^2$ square with $\ell^2 - 1$ rooks in it. This square can be partitioned into ℓ^2 squares of size $\ell \times \ell$, so one of them is empty.

Problem 3. Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

$$\angle CHS - \angle CSB = 90°, \quad \angle THC - \angle DTC = 90°.$$

Prove that line BD is tangent to the circumcircle of triangle TSH.

Solution. Let the line passing through C and perpendicular to the line SC intersect the line AB at Q (see Figure 1). Then

$$\angle SQC = 90° - \angle BSC = 180° - \angle SHC,$$

which implies that the points C, H, S, and Q lie on a common circle. Moreover, since SQ is a diameter of this circle, we infer that the circumcentre K of triangle SHC lies on the line AB. Similarly, we prove that the circumcentre L of triangle CHT lies on the line AD.

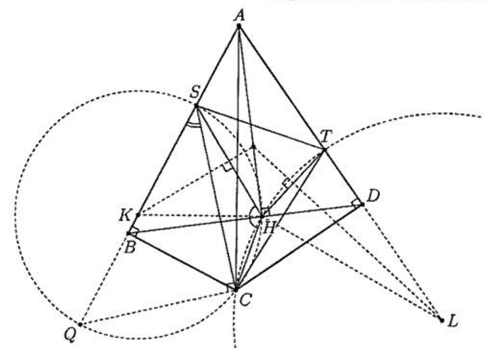


Figure 1

In order to prove that the circumcircle of triangle SHT is tangent to BD, it suffices to show that the perpendicular bisectors of HS and HT intersect on the line AH. However, these two perpendicular bisectors coincide with the angle bisectors of angles AKH and ALH. Therefore, in order to complete the solution, it is enough (by the bisector theorem) to show that

$$\frac{AK}{KH} = \frac{AL}{LH}. \tag{1}$$

We present two proofs of this equality.

First proof. Let the lines KL and HC intersect at M (see Figure 2). Since $KH = KC$ and $LH = LC$, the points H and C are symmetric to each other with respect to the line KL. Therefore M is the midpoint of HC. Denote by O the circumcentre of quadrilateral $ABCD$. Then O is the midpoint of AC. Therefore we have $OM \parallel AH$ and hence $OM \perp BD$. This together with the equality $OB = OD$ implies that OM is the perpendicular bisector of BD and therefore $BM = DM$.

Since $CM \perp KL$, the points B, C, M, and K lie on a common circle with diameter KC. Similarly, the points L, C, M, and D lie on a circle with diameter LC. Thus, using the sine law, we obtain

$$\frac{AK}{AL} = \frac{\sin\angle ALK}{\sin\angle AKL} = \frac{DM}{CL} \cdot \frac{CK}{BM} = \frac{CK}{CL} = \frac{KH}{LH},$$

which finishes the proof of (1).

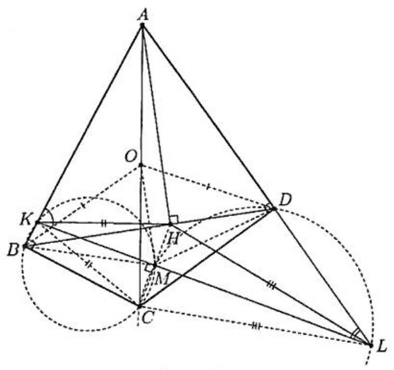


Figure 2

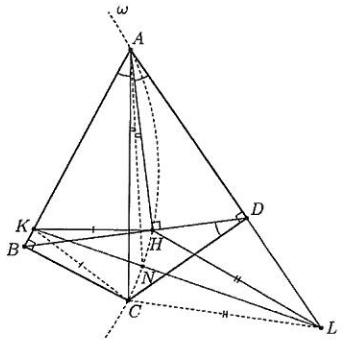


Figure 3

Second proof. If the points A, H, and C are collinear, then $AK = AL$ and $KH = LH$, so the equality (1) follows. Assume therefore that the points A, H, and C do not lie in a line and consider the circle ω passing through them (see Figure 3). Since the quadrilateral $ABCD$ is cyclic,

$$\angle BAC = \angle BDC = 90^\circ - \angle ADH = \angle HAD .$$

Let $N \neq A$ be the intersection point of the circle ω and the angle bisector of $\angle CAH$. Then AN is also the angle bisector of $\angle BAD$. Since H and C are symmetric to each other with respect to the line KL and $HN = NC$, it follows that both N and the centre of ω lie on the line KL. This means that the circle ω is an APOLLONIUS circle of the points K and L. This immediately yields (1).

Comment. Either proof can be used to obtain the following generalised result:

Let $ABCD$ be a convex quadrilateral and let H be a point in its interior with $\angle BAC = \angle DAH$. The points S and T are chosen on the sides AB and AD, respectively, in such a way that H lies inside triangle SCT and

$$\angle SHC - \angle BSC = 90^\circ, \quad \angle THC - \angle DTC = 90^\circ .$$

Then the circumcentre of triangle SHT lies on the line AH (and moreover the circumcentre of triangle SCT lies on AC).

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Problem 4. Points P and Q lie on side BC of an acute-angled triangle ABC so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines BM and CN intersect on the circumcircle of triangle ABC.

Solution 1. Denote by S the intersection point of the lines BM and CN. Let moreover $\beta = \angle QAC = \angle CBA$ and $\gamma = \angle PAB = \angle ACB$. From these equalities it follows that the triangles ABP and CAQ are similar (see Figure 1). Therefore we obtain

$$\frac{BP}{PM} = \frac{BP}{PA} = \frac{AQ}{QC} = \frac{NQ}{QC} .$$

Moreover,

$$\angle BPM = \beta + \gamma = \angle CQN .$$

Hence the triangles BPM and NQC are similar. This gives $\angle BMP = \angle NCQ$, so the triangles BPM and BSC are also similar. Thus we get

$$\angle CSB = \angle BPM = \beta + \gamma = 180^\circ - \angle BAC ,$$

which completes the solution.

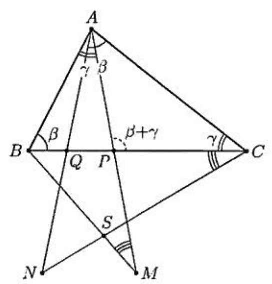

Figure 1

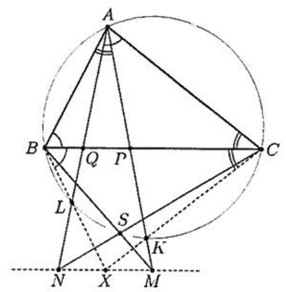

Figure 2

Solution 2. As in the previous solution, denote by S the intersection point of the lines BM and NC. Let moreover the circumcircle of the triangle ABC intersect the lines AP and AQ again at K and L, respectively (see Figure 2).

Note that $\angle LBC = \angle LAC = \angle CBA$ and similarly $\angle KCB = \angle KAB = \angle BCA$. It implies that the lines BL and CK meet at a point X, being symmetric to the point A with respect to the line BC. Since $AP = PM$ and $AQ = QN$, it follows that X lies on the line MN. Therefore, using PASCAL's theorem for the hexagon $ALBSCK$, we infer that S lies on the circumcircle of the triangle ABC, which finishes the proof.

Comment. Both solutions can be modified to obtain a more general result, with the equalities

$$AP = PM \quad \text{and} \quad AQ = QN$$

replaced by

$$\frac{AP}{PM} = \frac{QN}{AQ} .$$

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Problem 5. For each positive integer n, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

Solution. We will show that for every positive integer N, any finite collection of these coins of total value at most $N - \frac{1}{2}$, can be split into N groups, each of total value at most 1. The problem statement is a particular case for $N = 100$.

We start with some preparations. If several given coins together have a total value also of the form $\frac{1}{k}$ for a positive integer k, then we may merge them into one new coin. Clearly, if the resulting collection can be split in the required way then the initial collection can also be split.

After each such merging, the total number of coins decreases, thus at some moment we come to a situation when no more merging is possible. At this moment, for every even k there is at most one coin of value $\frac{1}{k}$ (otherwise two such coins may be merged), and for every odd $k > 1$ there are at most $k - 1$ coins of value $\frac{1}{k}$ (otherwise k such coins may also be merged).

Now, clearly, each coin of value 1 should form a single group; if there are d such coins then we may remove them from the collection and replace N by $N - d$. So from now on we may assume that there are no coins of value 1.

Finally, we may split all the coins in the following way. For each $k = 1, 2, \ldots, N$ we put all the coins of values $\frac{1}{2k-1}$ and $\frac{1}{2k}$ into a group G_k; the total value of G_k does not exceed

$$(2k-2) \cdot \frac{1}{2k-1} + \frac{1}{2k} < 1.$$

It remains to distribute the "small" coins of values which are less than $\frac{1}{2N}$; we will add them one by one. On each step, take any remaining small coin. The total value of coins in groups at this moment is at most $N - \frac{1}{2}$, so there exists a group of total value at most $\frac{1}{N}(N - \frac{1}{2}) = 1 - \frac{1}{2N}$; thus it is possible to put our small coin into this group. Acting so, we will finally distribute all the coins.

Comment 1. The algorithm may be modified, at least the step where one distributes the coins of values $\geq \frac{1}{2N}$. One different way is to put into G_k all the coins of values $\frac{1}{(2k-1)2^s}$ for all integer $s \geq 0$. One may easily see that their total value also does not exceed 1.

Comment 2. The original proposal also contained another part, suggesting to show that a required splitting may be impossible if the total value of coins is at most 100. There are many examples of such a collection, e.g. one may take 98 coins of value 1, one coin of value $\frac{1}{2}$, two coins of value $\frac{1}{3}$, and four coins of value $\frac{1}{5}$.

The Problem Selection Committee thought that this part is less suitable for the competition.

Problem 6. A set of lines in the plane is in *general position* if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its *finite regions*. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to colour at least $\sqrt{n}$ of the lines blue in such a way that none of its finite regions has a completely blue boundary.

Note: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant c.

Solution 1. (for $c = \sqrt{1/2}$) Let L be the given set of lines. Choose a maximal (by inclusion) subset $B \subseteq L$ such that when we colour the lines of B blue, no finite region has a completely blue boundary. Let $|B| = k$. We claim that $k \geq \sqrt{n/2}$.

Let us colour all the lines of $L \setminus B$ red. Call a point *blue* if it is the intersection of two blue lines. Then there are $\binom{k}{2}$ blue points.

Now consider any red line ℓ. By the maximality of B, there exists at least one finite region A whose only red side lies on ℓ. Since A has at least three sides, it must have at least one blue vertex. Let us take one such vertex and associate it to ℓ.

Since each blue point belongs to four regions (some of which may be unbounded), it is associated to at most four red lines. Thus the total number of red lines is at most $4\binom{k}{2}$. On the other hand, this number is $n - k$, so

$$n - k \leq 2k(k-1), \quad \text{thus} \quad n \leq 2k^2 - k \leq 2k^2,$$

and finally $k \geq \sqrt{n/2}$, which gives the desired result.

Solution 2. (for $c = \sqrt{2/3}$) We modify the proof of the first solution to show that one has in fact $k = |B| \geq \sqrt{2n/3}$.

Let us make weighted associations as follows. Let a region A whose only red side lies on ℓ have k vertices, so that $k-2$ of them are blue. We associate each of these blue vertices to ℓ, and put the weight $\frac{1}{k-2}$ on each such association. So the sum of the weights of all the associations is exactly $n - k$.

Now, one may check that among the four regions adjacent to a blue vertex v, at most two are triangles. This means that the sum of the weights of all associations involving v is at most $1 + 1 + \frac{1}{2} + \frac{1}{2} = 3$. This leads to the estimate

$$n - k \leq 3\binom{k}{2},$$

or

$$2n \leq 3k^2 - k < 3k^2,$$

which yields $k \geq \sqrt{2n/3}$.

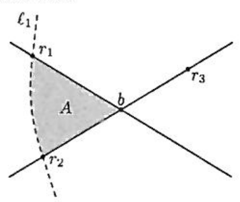
Solution 3. (for $c = 1$) Next, we even show that $k = |B| \geq \lceil \sqrt{n} \rceil$. For this, we specify the process of associating points to red lines in one more different way.

Call a point *red* if it lies on a red line as well as on a blue line. Consider any red line ℓ, and take an arbitrary finite region A whose only red side lies on ℓ. Let $r', r, b_1, \ldots, b_k$ be its vertices in clockwise order with $r', r \in \ell$; then the points r', r are red, while all the points b_1, $\ldots$, b_k are blue. Let us associate to ℓ the red point r and the blue point b_1. One may notice that to each pair of a red point r and a blue point b, at most one red line can be associated, since there is at most one region A having r and b as two clockwise consecutive vertices.

We claim now that at most two red lines are associated to each blue point b; this leads to the desired bound

$$n - k \leq 2\binom{k}{2} \quad \Longleftrightarrow \quad n \leq k^2.$$

Assume, to the contrary, that three red lines ℓ_1, ℓ_2, and ℓ_3 are associated to the same blue point b. Let r_1, r_2, and r_3 respectively be the red points associated to these lines; all these points are distinct. The point b defines four blue rays, and each point r_i is the red point closest to b on one of these rays. So we may assume that the points r_2 and r_3 lie on one blue line passing through b, while r_1 lies on the other one.



Now consider the region A used to associate r_1 and b with ℓ_1. Three of its clockwise consecutive vertices are r_1, b, and either r_2 or r_3 (say, r_2). Since A has only one red side, it can only be the triangle r_1br_2; but then both ℓ_1 and ℓ_2 pass through r_2, as well as some blue line. This is impossible by the problem assumptions.

Solution 4. (for $c = \sqrt{4/5}$) We assign weights as in Solution 2, but apply a different (weaker) estimate for the total weight associated to a point P. No two adjacent regions can be triangles in view of the condition that no three lines intersect in a common point. This leaves us with two possibilities: if only one of the four regions surrounding P is a triangle, the total weight is at most $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{5}{2}$. Otherwise, two opposite regions have to be triangles. The two red lines that border these triangles have to border the other two regions as well; hence these regions have two red sides each, so they do not contribute any weight at all. In this case, the total weight is 2. Thus we obtain the estimate

$$n - k \leq \frac{5}{2}\binom{k}{2},$$

which gives us $n \leq \frac{5}{4}k^2 - \frac{1}{4}k \leq \frac{5}{4}k^2$. So $k \geq \sqrt{4n/5}$, completing our proof.

Comment. There are examples of configurations with a maximal set of exactly $\sqrt{n}$ blue lines that cannot be extended further. Hence the "greedy" approach starting with a maximal set cannot be used to prove any bound better than $\sqrt{n}$.

Solution 5. (for $c = \sqrt{8/27}$) Let us colour each line blue with probability $p = \alpha/\sqrt{n}$ (α to be specified later). Then the expected number of blue lines is pn. On the other hand, it is well known that the number of finite regions is $(n-1)(n-2)/2$. Each of them is bounded by blue lines only with probability at most p^3. Therefore, the expected number of such regions is at most $p^3(n-1)(n-2)/2$. It suffices to discard one boundary line for each of these regions from the set of blue lines to obtain a feasible choice of blue lines. The expected number of blue lines remaining is

$$pn - \tfrac{p^3(n-1)(n-2)}{2} \geq pn - \tfrac{p^3n^2}{2} = \left(\alpha - \tfrac{\alpha^3}{2}\right)\sqrt{n}.$$

The maximum of the expression in brackets is $\sqrt{8/27}$, obtained for $\alpha = \sqrt{2/3}$. Since the expected value is $\geq \sqrt{8/27n}$ for this choice of α, there must be at least one feasible choice of at least $\sqrt{8/27n}$ lines.

Solution 6. (for $c = \frac{2}{3}$) The argument of the previous solution can be improved to $\frac{2}{3}$ (but no further without additional ideas). It is known that the number of triangles is at most $n(n-1)/3$, but it can be as large as $n(n-3)/3$, as shown in a paper by Füredi and Palásti ([2]). Thus we can improve the estimate above to

$$pn - \frac{p^3n(n-1)}{3} - \frac{p^4(n-1)(n-6)}{6} \geq \left(\alpha - \frac{\alpha^3}{3}\right)\sqrt{n} - \frac{\alpha^4}{6}.$$

Now the maximum of the expression in brackets is $\frac{2}{3}$, obtained for $\alpha = 1$, which finally shows that the expected number of blue lines remaining is at least $\frac{2}{3}\sqrt{n} - \frac{1}{6}$. Therefore, there must be at least one feasible choice with $\frac{2}{3}\sqrt{n} - \frac{1}{6}$ lines.

Comment. A very naïve application of the probabilistic method exhibited in the previous two solutions gives a bound of $\sqrt[3]{n}$: take $p = (3/(2n^2))^{1/3}$. Then the expected number of regions having only blue boundaries is less than $n^2p^3/2 = \frac{3}{4}$. Therefore, we obtain a feasible choice of blue lines with probability greater than $\frac{1}{4}$. On the other hand, the number of lines follows a binomial distribution and is therefore concentrated around its mean. Specifically, by Chebyshev's inequality from probability theory, the number of lines is less than $pn - n^{1/4}$ with probability at most $pn(n^{1/4})^{-2} = (3/2)^{1/3}n^{-1/6}$, which is less than $\frac{1}{4}$ for sufficiently large n. Thus there exists a feasible choice of at least

$$pn - n^{1/4} = (3/2)^{1/3}n^{1/3} - n^{1/4} > n^{1/3}$$

blue lines for sufficiently large n.

Comment. In spite of all the different estimates of order $\sqrt{n}$, this is not best possible. Using more advanced methods, it is possible to get a lower bound of the form $c\sqrt{n \log n}$ (as shown by USA leader Po-Shen Loh, based in particular on the results from a paper by Duke, Lefmann and Rödl ([1])).

References

[1] R.A. Duke, H. Lefmann and V. Rödl. On uncrowded hypergraphs, *Random Structures and Algorithms* **6** (1995), 209-212.

[2] Z. Füredi and I. Palásti. Arrangements of lines with a large number of triangles, *Proc. Amer. Math. Soc.* **92** (1984), 561-566.